Toward Interactive Statistical Modeling

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Workshop on Automated Program Generation for Computational Science
at ICCS 2010

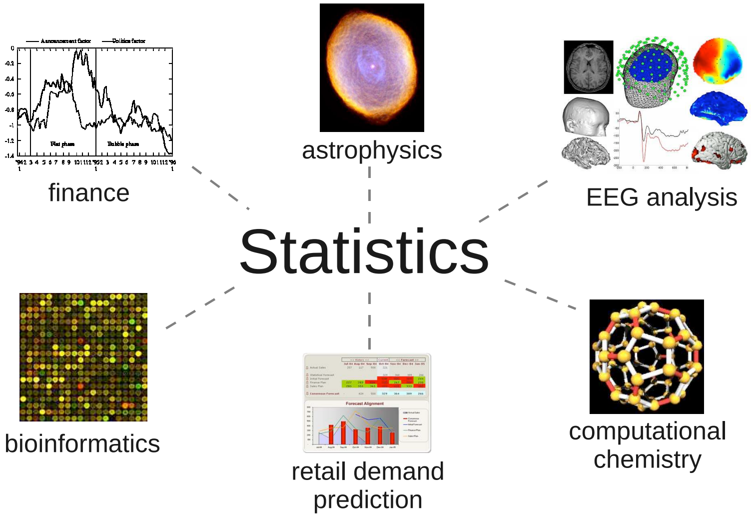
The team

Ashish Agarwal
Yale University
programming languages, optimization

Alexander Gray
Georgia Institute of Technology
machine learning, optimization

Richard Vuduc
Georgia Institute of Technology
high-performance computing, linear algebra

Statistics is everywhere



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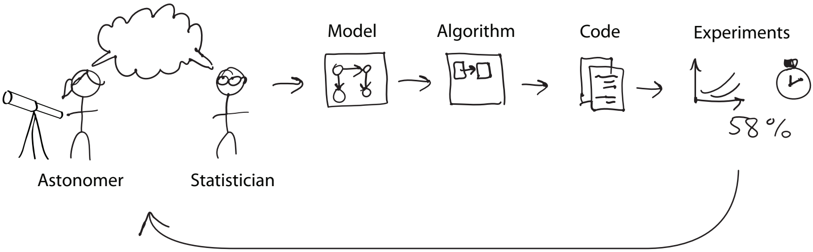

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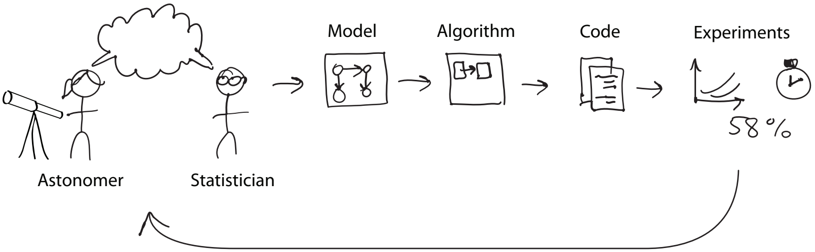

Vision: reduce development time while retaining correctness & efficiency.

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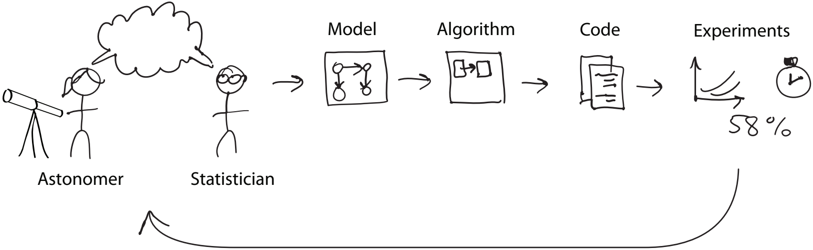

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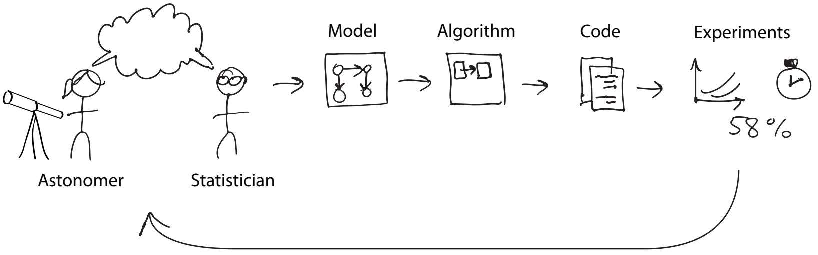

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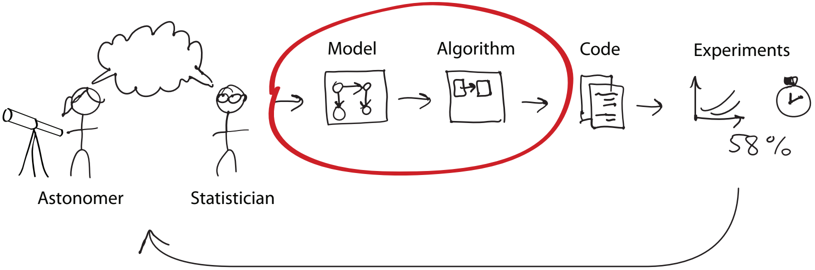

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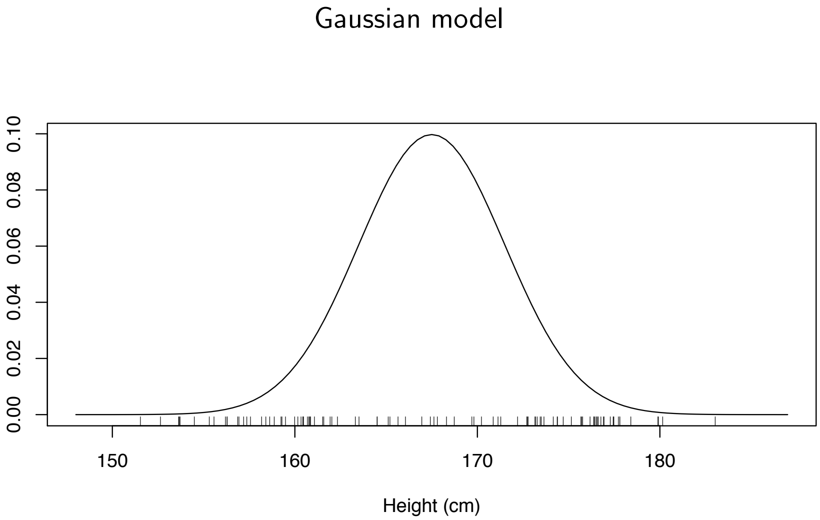

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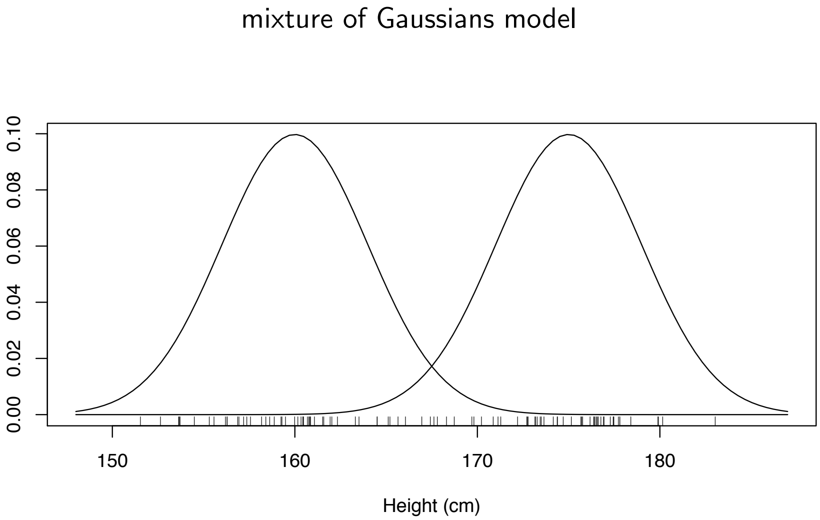

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Solution: *an EM algorithm (type of optimizer)*

```
(θ̂0, θ̂1) := rand();
while (...)
   for i = 1 to n do
      γi  :=  φ(xi; θ̂1,1)/(φ(xi; θ̂0,1)+φ(xi; θ̂1,1));
   θ̂0  :=  Σ_{i=1}^n (1-γi)*xi  /  Σ_{i=1}^n (1-γi);
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return (θ̂0, θ̂1);
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```
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   θ̂₀ := Σᵢ₌₁ⁿ (1-γᵢ)*xᵢ / Σᵢ₌₁ⁿ (1-γᵢ);
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return (θ̂₀, θ̂₁);
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- AutoBayes (2003) handles varying models for MLE/MAP – but there are many more possible axes of variation.

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Our approach

1. Rigorously defined mathematical language
    - enables stating statistical problems
2. *Schemas* – program transformations
    - embodiments of mathematical reformulations
3. Interactive algorithm assistant
    - enables exploring the space of correct solutions

Challenge 1: Statistics is big

Many types of mathematics

- probability, optimization, calculus, linear algebra, ...

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$\Longrightarrow$ Need an *expressive* symbolic representation of mathematics.

Challenge 2: Ensuring correctness

Expressive representation $\longrightarrow$ easier to create nonsensical expressions

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$\Longrightarrow$ We use type theory to promote correctness.

Type Theory 101

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1. Advanced type theories can be used to formalize mathematics.
    - an alternative to set theory
    - forms the basis of several modern theorem provers

2. These type theories have a computational interpretation.
    - Curry-Howard: type systems are logics
    - increased mathematical precision about programs

Language

language = syntax + type system + semantics

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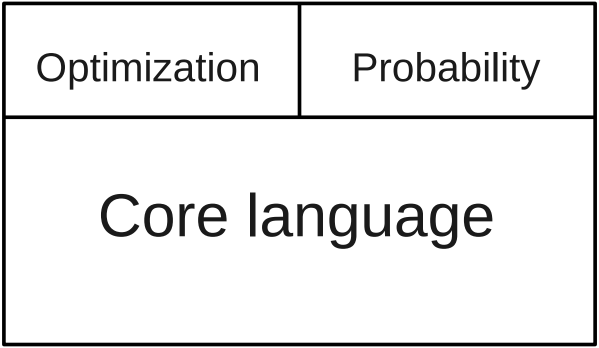

Language: Core

$$T ::= \texttt{Bool} \mid \texttt{Int} \mid \texttt{Real} \mid T_1 \times \ldots \times T_n \mid T_1 \to T_2$$

$$\begin{array}{llr} E ::= & \texttt{true} \mid \texttt{false} \mid \neg E \mid E_1 \vee E_2 \mid E_1 \wedge E_2 & \text{Booleans} \\ & \mid r \mid E_1 + E_2 \mid E_1 * E_2 \mid E_1^{E_2} \mid \log E & \text{reals} \\ & \mid (E_1, \ldots, E_n) \mid E.k & \text{tuples} \\ & \mid \texttt{if}\ E_1\ \texttt{then}\ E_2\ \texttt{else}\ E_3 \mid E_1 = E_2 \mid E_1 \leq E_2 & \text{conditions} \\ & \mid x \mid \lambda x : T \,.\, E \mid E_1\ E_2 \mid \texttt{fix}\ E & \text{functional core} \end{array}$$

- A typed lambda calculus + recursion.

Language: Optimization

$$E \mathrel{+}= \operatorname*{arg\,max}_{x_1 : T_1, \ldots, x_n : T_n} \{E_1 \mid E_2\} \qquad \text{optimization}$$

- Full account: "Automating Mathematical Program Transformations" (Agarwal et al., PADL 2010).

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  - We implement the necessary bookkeeping and restrictions.
- Functions must be measurable.
  - Non-measurable functions are not constructible in our language.

Language: Probability – new!

$$T \mathrel{+}= \texttt{Prob}\ T$$

$$\begin{array}{lr}
E \mathrel{+}= \texttt{bernoulli}\ E \mid \texttt{normal}\ E_1\ E_2 \mid \ldots & \text{distributions} \\
\quad \mid \texttt{return}\ E & \text{singleton distribution} \\
\quad \mid \texttt{var}\ x \sim E_1\ \texttt{in}\ E_2 & \text{random variables} \\
\quad \mid \texttt{condition}\ E_1\ \texttt{in}\ E_2 & \text{conditional distributions} \\
\quad \mid \mathbb{E}_{x \sim E_1}(E_2) & \text{expectation} \\
\quad \mid \texttt{pdf}\ E & \text{density functions}
\end{array}$$

- Probability monad (Giry 1981, Ramsey & Pfeffer 2003).
- **Continuous random variables!** (e.g., `Prob Real`)
- Semantics in terms of measure theory.

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- Mechanization requires a higher level of formalism.

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Logically: expression reformulation theorems

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Current schema library contains 100+ schemas

- computer algebra
- propositional logic
- equation manipulation
- calculus
- optimization
- probability & statistics

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Relies crucially on symbolic operations.

Interactive algorithm assistant

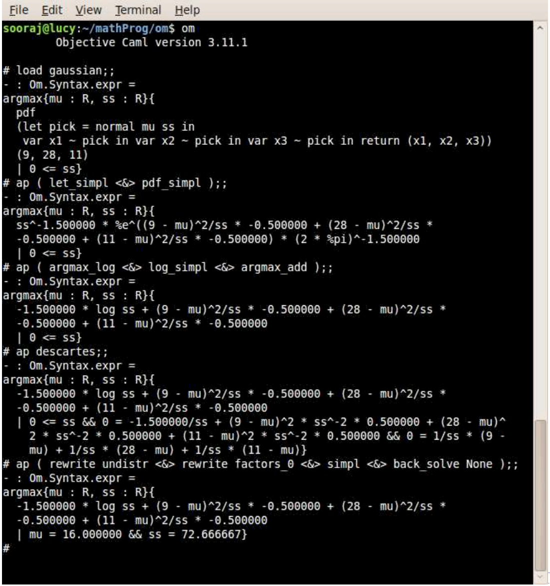
Features

- enter problems
- apply schemas
- undo/redo
- combinators

Status

- can solve several textbook examples of MLE, incl. via EM
- autotuning + more sophisticated code generation is planned

Come see me for a demo!



Conclusions

- The first symbolic formalization of statistics for freely expressing statistical problems and reformulations
  - type-theoretic formalization of probability & optimization
  - continuous probability distributions
- An implementation of the language & schemas
  - the Expectation-Maximization (EM) schema
  - interactive algorithm assistant
- Future plans include incorporating feedback
  - autotuning, high-performance code generation
  - model selection, causal inference

Thank you

Fin.